

MAS 117

Introduction to Algebra

Solutions 3

January 2007

1 If we take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then

$$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O_{2 \times 2}.$$

2 Let $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Then

$$\det(A - xI_2) = \begin{vmatrix} p-x & q \\ r & s-x \end{vmatrix} = x^2 - (p+s)x + (ps - qr),$$

so $a = -(p+s)$ and $b = ps - qr$. Then

$$A^2 + aA + bI_{2 \times 2} = \begin{pmatrix} p^2 + qr & pq + qs \\ rp + sr & sq + s^2 \end{pmatrix} - (p+s) \begin{pmatrix} p & q \\ r & s \end{pmatrix} + (ps - qr) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = O_{2 \times 2}.$$

3 (a) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Then

$$AB = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}, \quad BA = \begin{pmatrix} pa + qc & pb + qd \\ ra + sc & rb + sd \end{pmatrix},$$

so

$$\text{Trace}(AB) = ap + br + cq + ds = pa + qc + rb + sd = \text{Trace}(BA).$$

(b) There cannot exist two 2×2 matrices A and B such that $AB - BA = I_2$, since $\text{Trace}(AB - BA) = \text{Trace}(AB) - \text{Trace}(BA) = 0$ but $\text{Trace}(I_2) = 2$.

(c) To extend this to $n \times n$ matrices, let $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$. Then

$$AB = \left(\sum_{k=1}^n a_{ik} b_{kj} \right)_{n \times n},$$

so

$$\text{Trace}(AB) = \sum_{i=1}^n \sum_{k=1}^n a_{ik} b_{ki}.$$

(We put $j = i$ to get an entry on the diagonal, and then sum these entries to find the trace.)

Similarly

$$\text{Trace}(BA) = \sum_{i=1}^n \sum_{k=1}^n b_{ik} a_{ki}.$$

These two expressions are equal – to see this, remember that i and k are “dummy variables” in the summations, so we can change their names; if we exchange the names i and k , and then interchange the order of summation, the first expression becomes the second.

The proof that $AB - BA = I_n$ has no solution is the same as for $n = 2$ (using $\text{Trace}(I_n) = n$).

4 The fact that $A \triangle B = B \triangle A$ is clear from the definition – the elements lying in A or B but not both are the same as the elements in B or A but not both.

$(A \triangle B) \triangle C$ consists of the elements lying in either $A \triangle B$ or in C but not in both, which is the same as the set of elements lying either in all three of A, B, C or in just one of them. [For such an element is either in $A \triangle B$ (that is, in one of A and B but not both) and not in C , or else not in $A \triangle B$ (that is, in neither A nor B , or in both A and B), and in C .]

Clearly $A \triangle (B \triangle C)$ is the same.

The identity is $\emptyset$ since $A \triangle \emptyset = A$. (To say that an element lies in either A or $\emptyset$ but not both is the same as to say that it lies in A .) The inverse of A is A , since $A \triangle A = \emptyset$. (No element lies in either A or A but not both!)

For the distributive law, an element lying in $A \triangle B$ and also in C must lie in $A \cap C$ or $B \cap C$ but not both.

You may if you wish draw Venn diagrams to answer this part, but you must explain how the Venn diagrams corresponding to the two sides of the equation are produced.